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Numerical Solution of the Spinor Helmholtz Equation in Optics

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This paper discusses a method of solution for the spinor Helmholtz equation in optics as it was previously made [Hillion, *J. Computational Physics* **28** (1978)] for the scalar Helmholtz equation. A new approximation is given with the noteworthy property that the spinor field is reduced to a set of two scalar fields satisfying similar difference equations. And, as in the previous case, the corresponding explicit methods are stable provided that a rather unrestrictive condition is satisfied. An example of computation is given.

I. INTRODUCTION

As previously shown [1], a polarized laser beam propagating along the $0z$ axis is characterized by a complex spinor $\Psi(r)$ ($r = (x, y, z) \stackrel{d}{=} (\rho, z)$) which is a solution of the spinor Helmholtz equation

$$(\sigma^j \partial_j + iK_0(1 + \epsilon\mu(r))\, \Psi(r) = 0, \qquad i = (-1)^{1/2}, \tag{1}$$

where the σ^j, $j = 1, 2, 3$, are the Pauli matrices

$$\sigma^1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}; \qquad \sigma^2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}; \qquad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \tag{2}$$

satisfying the following relations

$$\sigma^l \sigma^j = i\epsilon^{ljk}\sigma^k, \qquad \sigma^l\sigma^j + \sigma^j\sigma^l = 2\delta^{lj}, \tag{2'}$$

where δ^{lj} and ϵ^{ljk} are, respectively, the Kronecker tensor and the permutation tensor. In Eq. (1) the ∂_j, $j = 1, 2, 3$, are the derivatives with respect to x, y, z; K_0 is the wave number, $n(r) = 1 + \epsilon\mu(r)$ denotes the index of refraction, the quantity ϵ measuring the deviation of the index from unity. We use the summation convention $\sigma^j\partial_j = \sigma^1\partial_1 + \sigma^2\partial_2 + \sigma^3\partial_3$.

We previously discussed [2] the properties of Eq. (1) and we showed that Eq. (1) has one and only one solution for such boundary conditions as

$$(\Psi(r))_{z=0} = \Psi^0(\rho) \text{ (1a)}; \qquad \Psi(z, \rho) = O(|\rho|^{-\alpha}) \qquad \text{as } \rho \mapsto \infty;\ \alpha \geqslant 2, \forall z \geqslant 0. \quad \text{(1b)}$$

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We intend here to give a numerical method for solving Eq. (1) in optics where K_0 is about $10^5\,\text{cm}^{-1}$ which leads to some difficulties already discussed in a previous paper [3] and, in the present work, we closely follow the method given in [3] beginning with the case of a field propagating in free space where $\epsilon = 0$ so that Eq. (1) reduces to

$$(\sigma^j \partial_j + iK_0)\,\Psi(r) = 0 \tag{3}$$

2. Paraxial Approximations to the Solutions of Eq. (3)

2.1. *Formal Introduction*

In this section, we introduce paraxial approximations in a formal and rather intuitive way while in Appendix 1, we give rigorous justifications and higher-order approximations.

First, we write (3) in the form

$$\partial_3 \Psi(r) = -i(L + \sigma_3 K_0)\,\Psi(r); \quad L = \sigma_2\partial_1 - \sigma_1\partial_2\,, \; \partial_1 = \frac{\partial}{\partial x}\,, \; \partial_2 = \frac{\partial}{\partial y}\,, \; \partial_3 = \frac{\partial}{\partial z}\,. \tag{3'}$$

The solutions of (3) are

$$\Psi(r) = \text{const}\exp\{-i(L + \sigma_3 K_0)\,z\}\,\Psi_0(\rho)$$

and they satisfy the recurrence relation ($\Psi^n = \Psi(\rho, nh)$),

$$\Psi^{n+1} = \exp\{-i(L + \sigma_3 K_0)\,h\}\,\Psi^n. \tag{4}$$

Now, we have (see Appendix 1), $(L + \sigma_3 K_0)^2 = \Delta_\perp + K_0^2$ so that

$$\begin{aligned} \cos\{(L + \sigma_3 K_0)\,h\} &= \cos((K_0^2 + \Delta_\perp)^{1/2}\,h), \qquad \Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,, \\ \sin\{(L + \sigma_3 K_0)\,h\} &= (L + \sigma_3 K_0)(K_0^2 + \Delta_\perp)^{-1/2}\sin((K_0^2 + \Delta_\perp)^{1/2}\,h). \end{aligned} \tag{4'}$$

These relations are easy to prove by comparison of power series expansions in both sides.

The paraxial approximation valid for slow-transverse variations on the wavelength scale and slow variation of K_0 is defined as the first-order approximation in $\Delta_\perp$ to the relations (4)

$$\cos((K_0^2 + \Delta_\perp)^{1/2}\,h) \simeq \cos K_0 h - \frac{h}{2K_0}\sin K_0 h\,\Delta_\perp)$$

$$\begin{aligned} &(L + \sigma_3 K_0)(K_0^2 + \Delta_\perp)^{-1/2}\sin((K_0^2 + \Delta_\perp)^{1/2}\,h) \\ &\quad \simeq \frac{(L + \sigma_3 K_0)}{K_0}\Big(\sin K_0 h + \frac{h}{2K_0}\Big(\cos K_0 h - \frac{\sin K_0 h}{K_0 h}\Big)\,\Delta_\perp\Big). \end{aligned}$$

Upon inserting these expressions into (4), one has

$$\Psi^{n+1} = \left\{\cos K_0 h - i\frac{\sin K_0 h}{K_0}(L + \sigma_3 K_0)\right\}\Psi^n$$
$$- \frac{h}{2K_0}\left\{\sin K_0 h + \frac{i}{K_0}\left(\cos K_0 h - \frac{\sin K_0 h}{K_0 h}\right)(L + \sigma_3 K_0)\right\}\Delta_\perp \Psi^n \qquad (5)$$

but one can still make further approximations:

(i) from the paraxial conditions

$$|K_0 \Psi(r)| \gg \sup\{|\partial x\, \Psi(r)|, |\partial y\, \Psi(r)|\}$$

one has $(L + \sigma_3 K_0)\Psi^n \simeq \sigma_3 K_0 \Psi^n$.

(ii) since the stepsize h is in the range 10–100 m, one has, $K_0 h \gg 1$ and

$$\cos K_0 h - \frac{\sin K_0 h}{K_0 h} \simeq \cos K_0 h.$$

Using these approximations in Eq. (5),

$$\Psi^{n+1} = e^{-i\sigma_3 K_0 h}\left(1 - i\sigma_3 \frac{h}{2K_0}\Delta_\perp\right)\Psi^n \qquad (5')$$

and with $\Delta_\perp$ replaced by the 5 point Poisson operator

$$\Psi_{jl}^{n+1} = e^{-i\sigma_3 K_0 h}\left[\Psi_{jl}^n - i\sigma_3 \frac{b}{2}(\Psi_{j+1,l}^n + \Psi_{j-1,l}^n + \Psi_{j,l+1}^n + \Psi_{j,l-1}^n - 4\Psi_{jl}^n)\right] \qquad (6)$$

with $b = h/K_0 h_\rho^2$, where $h\rho$ is the stepsize in both directions ox and oy and with $\Psi_{jl}^n = \Psi(jh\rho, lh\rho, nh)$.

From the definition (2) of the matrix σ_3, it appears that Eq. (6) is a set of two uncoupled scalar equations; this is an important result. In this paper, we focus on Eq. (6) and on its properties.

2.2. *Properties of Eq.* (6)

In this section, we discuss the local truncation error and the stability of Eq. (6) and we prove that there exists a constant of motion.

The local truncation error is easy to obtain since from Appendix 1 Eq. (6) is a power series in b truncated after the second term and since, as is well known, the 5-point Poisson operator in an approximation of the Laplacian to the order $O(h_\rho^2)$.

To test the stability of (6), one writes (5′)

$$\Psi^{n+1} = \left(\cos K_0 h - \frac{h}{2K_0}\sin K_0 h\, \Delta_\perp\right)\Psi^n - i\sigma_3\left(\sin K_0 h + \frac{h}{2K_0}\cos K_0 h\, \Delta_\perp\right)\Psi^n$$

so that $\Psi^{n+1} = \Psi_1^{n+1} + \Psi_2^{n+1}$ with

$$\Psi_1^{n+1} = \left(\cos K_0 h - \frac{h}{2K_0} \sin K_0 h \, \Delta_\perp\right) \Psi^n$$

$$\Psi_2^{n+1} = -i\sigma_3 \left(\sin K_0 h + \frac{h}{2K_0} \cos K_0 h \, \Delta_\perp\right) \Psi^n. \tag{7}$$

Using now the von Neumann harmonic analysis with solutions $\Psi_{jl}^n = e^{ij\xi} e^{il\eta} \Phi^n$, one obtains easily the amplification factors G_1, G_2, of Eqs. (7) since one has $\Delta_\perp \Psi_{jl}^n = -(4/h_\rho^2)(\sin^2(\xi/2) + \sin^2(h/2))\, \Phi^n$,

$$G_1 = \cos K_0 h + 2b \sin K_0 h \left(\sin^2 \frac{\xi}{2} + \sin^2 \frac{\eta}{2}\right);$$

$$G_2 = -i\sigma_3 \left[\sin K_0 h - 2b \cos K_0 h \left(\sin^2 \frac{\xi}{2} + \sin^2 \frac{\eta}{2}\right)\right]$$

and as $0 \leqslant \lambda = \sin^2(\xi/2) + \sin^2(\eta/2) \leqslant 2$ the stability conditions for Eqs. (7) and thus for Eq. (6) are

$$\sup_{0 \leqslant \lambda \leqslant 2} |\cos K_0 h + 2b\lambda \sin K_0 h| \leqslant 1; \qquad \sup_{0 \leqslant \lambda \leqslant 2} |\sin K_0 h - 2\lambda b \cos K_0 h| \leqslant 1. \tag{8}$$

If $\sin K_0 h \cos K_0 h > 0$, these conditions reduce to

$$|\cos K_0 h + 4b \sin K_0 h| \leqslant 1. \tag{8'}$$

As shown in Appendix 1, Eq. (5') corresponds to the first two terms of a power series in b which converges if $b < 1$. The conditions (8) with $b < 1$ are not very restrictive.

Now, we prove that there exists for Eq. (6) a constant of motion valid to the order $O(b^2)$; a more general result is given in Appendix 2. Since σ_3 is an Hermitian matrix, the adjoint (Hermitian conjugate) equation of (5') is

$$\Psi^{\dagger n+1} = \Psi^{\dagger n} \left(1 + i\sigma_3 \frac{h}{2K_0} \Delta_\perp\right) e^{i\sigma_3 K_0 h} \tag{5''}$$

so that, one has from (5') and (5'')

$$\Psi^{\dagger n+1} \Psi^{n+1} = \Psi^{\dagger n} \Psi^n + \frac{ih}{2K_0} (\Delta_\perp \Psi^{\dagger n} \cdot \Psi^n - \Psi^{\dagger n} \cdot \Delta_\perp \Psi^n) + O(b^2)$$

and

$$\int_{z=\text{const}} \Psi^{\dagger n+1} \Psi^{n+1} \, dx \, dy = \int_{z=\text{const}} \Psi^{\dagger n} \Psi^n \, dx \, dy$$

$$+ \frac{ih}{2K_0} \int_{z=\text{const}} (\Delta_\perp \Psi^{\dagger n} \cdot \Psi^n - \Psi^{\dagger n} \cdot \Delta_\perp \Psi_n) \, dx \, dy + O(b^2)$$

but the second integral on the right-hand side is zero. Just apply Green's theorem taking into account (1b), so one has

$$\int_{z=\text{const}} \Psi^{\dagger n+1}\Psi^{n+1}\, dx\, dy = \int_{z=\text{const}} \Psi^{\dagger n}\Psi^{n}\, dx\, dy + O(b^2) \tag{9}$$

but $(\Psi^{\dagger}(r)\,\Psi(r))_{z=\text{const}}$. is the transverse energy density so that Eq. (9) implies the conservation of the transverse energy. This relation is also interesting from a numerical point of view since it makes it possible to test the stability and the importance of round-off errors.

2.3. *Numerical Tests*

We used the difference equation (6) for investigating the propagation of a Gaussian laser beam of wavelength $10.6\mu m$ with the following data

$$K_0 = 5.927533310 \times 10^5\ \text{cm}^{-1}; \qquad \Psi^0(x, y) = \frac{1}{\pi^{1/2}}\, e^{-(x^2+y^2)/a^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad a = 10\ \text{cm}.$$

We noticed previously that (6) is a set of two uncoupled scalar equations and besides, with the real boundary data $\Psi^0(x, y)$, the solutions of these equations are complex conjugate; so, one has just to compute the first component ψ_{jl}^n of Ψ_{jl}^n, since for the second one φ_{jl}^n one has $\varphi_{jl}^n = \bar{\psi}_{jl}^n$.

The computations of ψ_{jl}^n were made up to 5 km with a traverse stepsize $h\rho = 3$ cm, so one has with $h = 50$ m

$$\cos K_0 h = 0.67498; \qquad \sin K_0 h = 0.73783; \qquad b = 0.0937;$$
$$\cos K_0 h + 4b \sin K_0 h = 0.95152.$$

Tables I, II, III, in Appendix 3 contain the results, respectively, for 0, 1, 2, and 5 km and since the problem has a cylindrical symmetry around the 0 axis, one has just to give the values of $\operatorname{Re} \psi(r)$ and $\operatorname{Im} \psi(r)$ on a square grid, in one quadrant of the transverse plane the dimension of the useful square being 15×15 cm. The values of the energy density $\bar{\psi}(r)\, \psi(r)$ are also shown and one can see that the integral (9) is constant with a good precision, the exact value being 10^{-2}.

We checked the condition (8′) by changing $h\rho = 3$ cm into $h\rho = 1.5$ cm so that $|\cos K_0 h + 4b \sin K_0 h| = 1.781$ and actually a strong instability appeared about 1.2 km.

Since both components of $\Psi(r)$ are uncoupled and since they both satisfy the scalar Helmholtz equation [2], it is interesting to compare Tables I, II, and III of this paper with Tables I, II, and III of [3]. Of course the values of $\operatorname{Re} \psi$ and $\operatorname{Im} \psi$ differ but one can see that the results for the energy density $\bar{\psi}(r)\, \psi(r)$ are in good agreement.

Remark. The boundary condition used in this computation corresponds to a completely polarized beam with a linear (first and third quadrants) polarization [4] because the Stokes parameters are $\Psi^{\dagger}(r)\, \sigma_\mu \Psi(r)$, $\mu = 0, 1, 2, 3$, $\sigma_\mu = \{\sigma_0, \sigma_j\}$.

3. Paraxial Approximation to the Solutions of Eq. (1)

One now considers the solution of Eq. (1) where instead of K_0, one has $K = K_0 (1 + \epsilon\mu(r))$ and, as in [3], we have to make supplementary assumptions on $\mu(r)$ to find a convenient method of solution. The following assumptions seem rather unrestrictive in optics

$$|K_0\mu^l(r)| \gg \sup\{|\partial x\, \mu^l(r)|, |\partial y\, \mu^l(r)|\} \text{ (a)}; \quad |K_0{}^2\mu^l(r)| \gg |\Delta_\perp \mu^l(r)| \text{ (b)} \quad l = 1, 2, \ldots \tag{10}$$

One proves easily that (10a) implies that $(L + \sigma_3 K)^2 \simeq \Delta_\perp + K^2$ neglecting $L\mu(r)$ and it was shown in [3], that (10b) implies:

$$(\Delta_\perp + K^2)^p \simeq \sum_{j=0}^{p} \binom{p}{j} K_0^{2(p-j)} \Delta_\perp{}^j$$

As a consequence, the approximations to the solution of Eq. (1) are deduced from the approximations to the solution of Eq. (2) (see Appendix 1) only changing K_0 into $K_0 (1 + \epsilon\mu(r))$.

In particular for the paraxial approximation (notice that for this approximation, the inequality (10b) is unnecessary) Eq. (6) becomes

$$\Psi_{jl}^{n+1} = e^{-i\sigma_3 K_{jl}^n h}\left[\Psi_{jl}^n - i\sigma_3 \frac{b_{jl}}{2} (\Psi_{j+1,l}^n + \Psi_{j-1,l}^n + \Psi_{j,l+1}^n + \Psi_{j,l-1}^n - 4\Psi_{j,l}^n)\right] \tag{11}$$

with

$$K_{j,l}^n = K_0(1 + \epsilon\mu(jh_\rho, lh_\rho, nh))^{1/2}; \qquad b_{jl}^n = \frac{h}{K_{jl}^n h_\rho{}^2}. \tag{11'}$$

Remark. As Eq. (6), Eq. (11) is a set of two uncoupled scalar equations except when $\mu(r)$ results from a nonlinear interaction, for instance, with $\mu(r) = f(\Psi^\dagger(r)\,\Psi(r))$ then one has to solve simultaneously both Eqs. (11). Moreover, since stability for nonlinear interactions is not warranted, one must check at each step, the value of the transverse energy.

4. Discussion

We intended here to give a numerical method to describe in optics the propagation of a polarized beam in the frame of a spinor formalism whose mathematical properties were discussed in [2] while some physical implications were given in [5].

In linear optics, one can use the interesting property of the approximation (6) to reduce the two scalar equations, explicitly

$$(\Psi_{jl}^{n+1})_1 = e^{-iK_0 h}\left\{\Psi_{jl}^n - \frac{ib}{2} (\Psi_{j+1,l}^n + \Psi_{j-1,l}^n + \Psi_{j,l+1}^n + \Psi_{j,l-1}^n - 4\Psi_{jl}^n)\right\}_1, \tag{12a}$$

$$(\Psi_{jl}^{n+1})_2 = e^{iK_0 h}\left\{\Psi_{jl}^n + \frac{ib}{2} (\Psi_{j+1,l}^n + \Psi_{j-1,l}^n + \Psi_{j,l+1}^n + \Psi_{j,l-1}^n - 4\Psi_{jl}^n)\right\}_2, \tag{12b}$$

where $(\Psi_{jl}^n)_1$, $(\Psi_{jl}^n)_2$ are both components of Ψ_{jl}^n. These equations are very similar to that obtained in the case of the scalar Helmholtz equation [3]

$$\psi_{jl}^{n+1} + \psi_{jl}^{n-1} = 2\cos(K_0 h)\, \psi_{jl}^n - b \sin(K_0 h)(\psi_{j+1,l}^n + \psi_{j-1,l}^n + \psi_{j,l+1}^n + \psi_{j,l-1}^n - 4\psi_{jl}^n). \tag{13}$$

From a numerical point of view, the comparison between expressions (12) and (13) is interesting; indeed, if the boundary data are chosen such that for one component, for instance, $(\Psi_{jl}^n)_1$, one has $|(\Psi_{jl}^0)_1|^2 = |\psi_{jl}^0|^2$ then it solving (13) or (12a) is immaterial, as long as one is only interested in $|(\Psi_{jl}^n)_1|^2$ since $|(\Psi_{jl}^n)_1|^2 = |\psi_{jl}^n|^2$. This can be seen on the results in Appendix 3 of this paper and on those in Appendix 3 of [3].

But, from a physical point of view, one has to solve *successively* both equations (12) to compute the power density of the beam $|(\Psi_{jl}^n)_1|^2 + |(\Psi_{jl}^n)_2|^2$ and polarization and the Poynting quadrivector $\Psi^\dagger(r)\,\sigma_\mu \Psi(r)$, $\sigma_\mu = \{\sigma_0, \sigma_i\}$, $i = 1, 2, 3$, where σ_0 is the 2×2 identity matrix, (the σ_i's are the Pauli matrices and $\Psi^\dagger(r)$ is the Hermitian conjugate field).

Besides, if we note

$$(\Phi_{jl}^n)_1 = e^{iK_0 nh}(\Psi_{jl}^n)_1, \qquad (\Phi_{jl}^n)_2 = e^{-iK_0 nh}(\Psi_{jl}^n)_2,$$

one deduces from (12), the following equations

$$\begin{aligned}(\Phi_{jl}^n)_1 &= \left\{\Phi_{jl}^n - \frac{ib}{2}(\Phi_{j+1,l}^n + \Phi_{j-1,l}^n + \Phi_{j,l+1}^n + \Phi_{j,l-1}^n - 4\Phi_{jl}^n)\right\}_1,\\ (\Phi_{jl}^n)_2 &= \left\{\Phi_{jl}^n + \frac{ib}{2}(\Phi_{j+1,l}^n + \Phi_{j-1,l}^n + \Phi_{j,l+1}^n + \Phi_{j,l-1}^n - 4\Phi_{jl}^n)\right\}_2,\end{aligned} \tag{14}$$

which are finite-difference approximations of the Schrödinger equations

$$\pm 2iK_0 \frac{\partial \varphi(r)}{\partial z} + \Delta_\perp \varphi(r) = 0 \tag{15}$$

regularly used in the frame of the paraxial approximation so that one could also solve (15) by the well-known methods. But, in our opinion, the approximations (12) and (13) are of much most interest since they correspond to an explicit but nevertheless stable algorithm allowing large stepsizes h because the important parameter is $b = h/K_0 h_\rho^2$ so that the condition $b < 1$ is easy to fulfill with the large values of K_0 in optics. Besides, the approximation (13) leads to some interesting physical implications discussed elsewhere [6].

In nonlinear optics, for instance, when the refractive index depends on the power density of the beam $n(\Psi^\dagger \Psi)$, one has to solve *simultaneously* for polarized beams both Eqs. (11) (which are the generalization of (6) to the case of a medium with variable refractive index). So, for nonlinear optics, the numerical algorithm is slightly more intrincate.

It does not seem that another formalism was ever proposed to study the propagation of a polarized beam at the Fresnel approximation although the polarization variations could, for instance, intervene in some suggested communication systems as "the polarization pulse code modulation system" or "the dual polarization system" with a direct effect on noise and crosstalk.

Appendix 1

First we prove the following relation ($L = \sigma_2 - \partial_1 - \sigma_1\partial_2$)

$$(L + \sigma_3 K_0)^2 = \Delta_\perp + K_0^2; \qquad \Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{A.1}$$

Indeed, one has

$$(L + \sigma_3 K_0)^2 = L^2 + (L\sigma_3 + \sigma_3 L)\, K_0 + K_0^2$$

and using (2′)

$$L^2 = (\sigma_2\partial_1 - \sigma_1\partial_2)^2 = \partial_1^2 + \partial_2^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_1)\,\partial_1\partial_2 = \partial_1^2 + \partial_2^2 = \Delta_\perp$$

and

$$L\sigma_3 + \sigma_3 L = (\sigma_2\sigma_3 + \sigma_3\sigma_2)\,\partial_1 - (\sigma_3\sigma_1 + \sigma_1\sigma_3)\,\partial_2 = 0$$

which proves (A.1).

Assuming now that $\Psi(r)$ is as smooth as is necessary to ensure the validity of all our expansions, one deduces from (4) and (A.1)

$$\Psi^{n+1} + \Psi^{n-1} = 2\sum_{p=0}^{\infty} (-1)^p \frac{h^{2p}}{(2p)!} (L + \sigma_3 K_0)^{2p}\, \Psi^n$$

$$= 2\sum_{p=0}^{\infty} (-1)^p \frac{h^{2p}}{(2p)!} (\Delta_\perp + K_0^2)^p\, \Psi^n$$

and from Ref. [3] ($x = K_0 h$)

$$\Psi^{n+1} + \Psi^{n-1} = 2\sum_{l=0}^{\infty} a_l \Delta_\perp{}^l \Psi^n; \qquad a_l = -\frac{h^{2l}}{2(l!)} \frac{d^{l-1}}{dx^{2(l-1)}} \frac{\sin x}{x} = \frac{(-1)^l}{2^l l!} \frac{h^{l+1}}{K_0^{l-1}} j_{l-1}(x),$$

$$l = 1, 2, \ldots, \tag{A.2}$$

using the spherical Bessel functions of the first kind $j_l(x)$. In the same way, one has

$$\Psi^{n+1} - \Psi^{n-1} = -2i \sum_{p=0}^{\infty} (-1)^p \frac{h^{2p+1}}{(2p+1)!} (L + \sigma_3 K_0)^{2p+1}\, \Psi^n$$

$$= -2i \sum_{p=0}^{\infty} (-1)^p \frac{h^{2p+1}}{(2p+1)!} (\Delta_\perp + K_0^2)^p\, \Phi^n$$

with

$$\Phi^n = (L + \sigma_3 K_0)\,\Psi^n$$

and rearranging the terms with respect to increasing powers of $\Delta_\perp{}^l \Phi^n$

$$\Psi^{n+1} - \Psi^{n-1} = -2i \sum_{l=0}^{\infty} h^{2l+1} \sum_{p=l}^{\infty} \frac{(-1)^p}{(2p+1)!} \binom{p}{l} (K_0 h)^{2(p-l)} \Delta_\perp{}^l \Phi^n$$

$$= -2i \sum_{l=0}^{\infty} \alpha_l \Delta_\perp{}^l \Phi^n \tag{A.3}$$

with

$$\alpha_l = h^{2l+1} \sum_{p=l}^{\infty} \frac{(-1)^p}{(2p+1)!} \binom{p}{l} (K_0 h)^{2(p-l)}. \tag{A.3'}$$

LEMMA 1. *For every nonnegative integer l, one has*

$$\alpha_l = \frac{h^{2l+1}}{l!} \frac{d^l}{dx^{2l}} \frac{\sin x}{x} = \frac{(-1)^l h^{l+1}}{l!\, 2^l K_0{}^l} j_l(x), \qquad l = 0, 1, 2,\ldots, \quad x = K_0 h. \tag{A.4}$$

In fact, with $p - l = j$, one can write (A.3′)

$$\alpha_l = \frac{h^{2l+1}}{l!} (-1)^j \sum_{j=0}^{\infty} (-1)^j \frac{(j+1)(j+2)\cdots(j+l)}{(2j+2l+1)!} x^{2j}$$

$$= \frac{h^{2l+1}}{l!} (-1)^l \frac{d^l}{dx^{2l}} \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+2l}}{(2j+2l+1)!}$$

$$= \frac{h^{2l+1}}{l!} \frac{d^l}{dx^{2l}} \left(\frac{1}{x} \sum_{j=0}^{\infty} (-1)^{j+l} \frac{x^{2l+2j+1}}{(2l+2j+1)!} \right.$$

$$= \frac{h^{2l+1}}{l!} \frac{d^l}{dx^{2l}} \left\{ \frac{1}{x} \left(\sin x - \sum_{K=1}^{l} (-1)^K \frac{x^{2K-1}}{(2K-1)!} \right) \right\}$$

$$= \frac{h^{2l+1}}{l!} \frac{d^l}{dx^{2l}} \frac{\sin x}{x}.$$

This completes the proof for the first part of the lemma. The second part is then easy from the definition of the Bessel functions.

COROLLARY. *The comparison between* (A.2) *and* (A.4) *gives*

$$\alpha_l = -2h \frac{\partial}{\partial x^2} a_l \, ; \qquad l = 0, 1, 2, \ldots \tag{A.4}$$

This relation makes easy the computations of the α_l *from the* a_l.

In summation one has

$$\Psi^{n+1} = \sum_{l=0}^{\infty} [a_l - i\alpha_l(L + \sigma_3 K_0)] \, \Delta_\perp^l \Psi^n \tag{A.5}$$

with a_l and α_l given, respectively, by (A.2) and (A.4). In optics and for sufficiently smooth fields, one has

$$(K_0 \Psi(r)| \gg \sup\{| \partial x \, \Psi(r)| \, , \, | \partial y \, \Psi(r)|\},$$

that is, $| \sigma_3 K_0 \Psi^n | \gg | L\Psi^n |$; so, Eq. (A.5) reduces to

$$\Psi^{n+1} = \sum_{l=0}^{\infty} (a_l - i\alpha_l \sigma_3 K_0) \, \Delta_\perp{}^l \Psi^n \tag{A.5$'$}$$

which is a system of two scalar equations.

We can now discuss the convergence of (A.5′). As in [3], one introduces the dimensionless quantities ξ, η, b.

$$x = h\rho \, \xi, \qquad y = h\rho \, \eta, \qquad b = h/K_0 h_\rho{}^2,$$

so that

$$\Delta_\perp \Psi^n = \frac{1}{h_\rho{}^2} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \Psi^{nd} = \frac{1}{h_\rho{}^2} \tilde{\Delta}_\perp \Psi^n.$$

In [3], we proved, the following result:

$$a_l = O(h^l/K_0{}^l), \qquad a_l \Delta_\perp{}^l = O(b^l) \tag{A.6}$$

so, from (A.6) and (A.6′), one has

$$\alpha_l = O(h^l/K_0^{l+1}), \qquad \alpha_l \sigma_3 K_0 \, \Delta_\perp{}^l = O(b^l). \tag{A.6$'$}$$

Then, it follows from (A.6) and (A.6′) that if $\tilde{\Delta}_\perp{}^l \Psi^n$ is bounded for every l the series (A.5′) converges for $b < 1$. The paraxial approximation (6) corresponds to the first two terms of (A.5′), the next higher approximation is

$$\Psi^{n+1} = e^{-i_3 K_0 h} \left(1 - \frac{ib}{2} \sigma^3 \tilde{\Delta}_\perp - \frac{b^2}{8} \tilde{\Delta}_\perp{}^2 \right) \Psi^n + O(b^3).$$

APPENDIX 2

We shall prove here that for the problem defined by Eqs. (1), (1a), and (1b), there exists a constant of motion.

Let Ω be a closed domain in a transverse plane and Γ be its boundary, $\Psi^{+}(r)$ being the Hermitian conjugate spinor, the Gauss–Ostrogradski theorem gives

$$\int_{\Omega} \partial^{\alpha}(\Psi^{\dagger}_{(r)}\, \sigma_{\alpha} \Psi_{(r)})\, dx\, dy = \int_{\Gamma} n^{\alpha} \Psi^{\dagger}_{(r)} \sigma_{\alpha} \Psi_{(r)}\, dl, \qquad \alpha = 1, 2,$$

where dl is the length element on Γ and n^{α} the normal Γ. Now, if Γ tends to infinity in both directions, the right-hand side of the previous relation is zero from (1b), so that on the transverse plane $z = \text{const.}$, one has

$$\int_{z=\text{const}} \partial^{\alpha}(\Psi^{\dagger}_{(r)} \sigma_{\alpha} \Psi_{(r)})\, dx\, dy = 0. \tag{A.7}$$

In (A.7), $\partial^{\alpha}\sigma_{\alpha} = \partial^{1}\sigma_{1} + \partial^{2}\sigma_{2}$, then it follows from (1) and from the adjoint equation

$$\Psi^{+}(r)(\sigma^{j}\overleftarrow{\partial}_{j} - iK_{0}(1 + \epsilon\mu(r)|) = 0 \qquad \text{(assuming } \mu(r) \text{ real)}$$

that

$$\partial^{\alpha}(\Psi^{+}(r)\, \sigma_{\alpha} \Psi(r)) = -\partial^{3}(\Psi^{+}(r)\, \sigma_{3} \Psi(r)).$$

Then, Eq. (A.7) gives

$$\int_{z=\text{const}} \Psi^{\dagger}_{(r)} \sigma_{3} \Psi_{(r)}\, dx\, dy = \text{const.}, \tag{A.8}$$

that is, provided that $\mu(r)$ is a real function, polarization is conserved during propagation.

APPENDIX 3

TABLE I

Solution of the Spinor Helmholtz Equation ($D = 100$ m)[a]

y (cm)	x (cm) 0	3	6	9	12	15
			Re ψ			
0	0.04919	0.04663	0.03972	0.03034	0.02073	0.01260
3	0.04663	0.04421	0.03764	0.02875	0.01963	0.01192
6	0.03972	0.03764	0.03203	0.02443	0.01665	0.01008
9	0.03034	0.02875	0.02443	0.01859	0.01263	0.0760
12	0.02073	0.01963	0.01665	0.01263	0.008526	0.005089
15	0.01260	0.01192	0.01008	0.00760	0.005089	0.002996
			Im ψ			
0	−0.5620	−0.5373	−0.04696	−0.3751	−0.2738	−0.1827
3	−0.5373	−0.5137	−0.4489	−0.3586	−0.2618	−0.1747
6	−0.4696	−0.4489	−0.3923	−0.3134	−0.2288	−0.1527
9	−0.3751	−0.3586	−0.3134	−0.2503	−0.1827	−0.1219
12	−0.2738	−0.2618	−0.2288	−0.1827	−0.1334	−0.08903
15	−0.1827	−0.1747	−0.1527	−0.1219	−0.08903	−0.05941
			$\bar{\psi}\psi$			
0	0.31826	0.29087	0.22210	0.14162	0.07540	0.03754
3	0.29087	0.26584	0.20293	0.12942	0.06892	0.03066
6	0.22210	0.20293	0.15493	0.09882	0.05263	0.02342
9	0.14162	0.12942	0.09882	0.06300	0.03354	0.01492
12	0.07540	0.06892	0.05263	0.03354	0.01787	0.00795
15	0.03354	0.03066	0.02342	0.01492	0.00795	0.00354

[a] $\int \bar{\psi}\psi\, dxdy = 0.99999 \times 10^{-2}$.

TABLE II

Solution of the Spinor Helmholtz Equation ($D = 2000$ m)[a]

y (cm)	x (cm) 0	3	6	9	12	15
			Re ψ			
0	−0.08910	−0.7867	−0.05131	−0.01686	0.01370	0.03289
3	−0.07867	−0.06892	−0.04339	−0.01139	0.01674	0.03402
6	−0.05131	−0.04339	−0.02280	0.002638	0.02431	0.03655
9	−0.01686	−0.01139	0.002638	0.01942	0.03266	0.03847
12	0.01370	0.01674	0.02431	0.03266	0.03788	0.03783
15	0.03289	0.03402	0.03655	0.03847	0.03783	0.03386
			Im ψ			
0	0.5304	0.5102	0.4536	0.3717	0.2794	0.1912
3	0.5102	0.4907	0.4360	0.3570	0.2680	0.1831
6	0.4536	0.4360	0.3869	0.3159	0.2632	0.1606
9	0.3717	0.3570	0.3159	0.2568	0.1908	0.1287
12	0.2794	0.2680	0.2362	0.1908	0.1405	0.09361
15	0.1912	0.1831	0.1606	0.1287	0.09361	0.06133
			$\bar{\psi}\psi$			
0	0.28926	0.26649	0.20839	0.13845	0.07825	0.03764
3	0.26649	0.24554	0.19198	0.12758	0.07210	0.03468
6	0.20839	0.19198	0.15024	0.09980	0.05638	0.02713
9	0.13845	0.12758	0.09980	0.06632	0.03747	0.01804
12	0.07825	0.07210	0.05638	0.03747	0.02118	0.01019
15	0.03764	0.03468	0.02713	0.01804	0.01019	0.00491

[a] $\int \bar{\psi}\psi \, dxdy = 0.10013 \times 10^{-1}$.

TABLE III

Solution of the Spinor Helmholtz Equation ($D = 5000\ m$)[a]

y (cm)	x (cm) 0	3	6	9	12	15
			Re ψ			
0	0.1976	0.2009	0.2086	0.2141	0.2101	0.1919
3	0.2009	0.2039	0.2103	0.2143	0.2087	0.1892
6	0.2086	0.2103	0.2137	0.2133	0.2035	0.1808
9	0.2141	0.2143	0.2133	0.2072	0.1919	0.1655
12	0.2101	0.2087	0.2035	0.1919	0.1718	0.1428
15	0.1919	0.1892	0.1808	0.1655	0.1428	0.1138
			Im ψ			
0	0.3892	0.3744	0.3320	0.2677	0.1912	0.1141
3	0.3744	0.3599	0.3184	0.2558	0.1814	0.1068
6	0.3320	0.3184	0.2798	0.2219	0.1538	0.08636
9	0.2677	0.2558	0.2219	0.1717	0.1135	0.05713
12	0.1912	0.1814	0.1538	0.1135	0.06784	0.02520
15	0.1141	0.1068	0.08636	0.05713	0.02520	−0.003030
			$\bar{\psi}\psi$			
0	0.19052	0.18054	0.15374	0.11750	0.08070	0.04984
3	0.18054	0.12953	0.14560	0.11136	0.07646	0.04720
6	0.15374	0.14560	0.12396	0.09474	0.06507	0.04015
9	0.11750	0.11136	0.09474	0.07241	0.04971	0.03066
12	0.08070	0.07646	0.06507	0.04971	0.03412	0.02103
15	0.04984	0.04720	0.04015	0.03066	0.02103	0.01296

[a] $\int \bar{\psi}\psi \, dxdy = 0.10034 \times 10^{-1}$.

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